Abstract

The problem of optimum data quantization for memoryless signal-detection systems operating in m-dependent noise environments is considered. The case where quantizer breakpoints are fixed or predetermined is considered first, and, for this case, existing results for general (unquantized) memoryless detection are modified to yield necessary and sufficient conditions for quantizer optimization in terms of asymptotic efficiency. Necessary conditions are also established for the optimal (asymptotically efficient) selection of quantizer breakpoints, and expressions are presented for the comparison of quantizer-detector performance on the basis of asymptotic relative efficiency.

1. Introduction

Signal detection systems frequently involve memoryless nonlinear operations on input data, and there are a number of practical advantages to replacing these detection nonlinearities with quantizers. Several authors have considered the problem of optimally designing quantizers for this purpose, and design criteria have been developed for quantizers to be used in a variety of signal and noise situations [1-3]. Many of these previous studies are based primarily on the assumption of independent sampling, and it is the purpose of this paper to extend some of these previous results to the case where there is dependence among the observed data samples.

To study this problem we consider the large-sample-size case of detecting known constant signals in additive noise, and model the dependence structure by assuming that the noise process is m-dependent. In general, an optimal detection procedure for this situation will require a memory of length m; however, we will restrict our study to those detectors which may be implemented without memory. The design of general (unquantized) memoryless detectors for this situation has been considered in [4] and here we apply the techniques of this earlier study to the corresponding quantizer design problem.

In Section 2 we state more precisely the problem to be considered and review previous pertinent results concerning general memoryless detectors and optimum quantization from [1]. Section 3 includes the derivation of design equations for optimum quantizer design for the case where the breakpoints are fixed, as well as optimization for the general quantizer case; and expressions are presented here for the comparison of detector performance for large sample sizes. In Section 4, four-level quantization is considered in detail for two particular m-dependent noise processes, and the performance of resulting quantizer designs is compared to that of previous designs for the corresponding case where independence was assumed.

2. Preliminaries

2.1. Problem Statement and Assumptions

As noted above, we consider the detection of a known constant signal in additive m-dependent noise. Specifically, we assume that we have a sequence \( y = (x_1, i=1, \ldots, n) \) of real observations of a process \( y = (x_1, i=1, \ldots, n) \); and, on the basis of \( y \), we wish to decide between the following pair of hypotheses concerning \( y \):

\[
H_0: x_i = N_i \quad ; \quad i = 1, \ldots, n
\]

versus

\[
H_1: x_i = N_i + \theta \quad ; \quad i = 1, \ldots, n
\]

where \( [N_i]_n \) is a zero-mean second-order-stationary m-dependent noise process and \( \theta \) is a known positive constant signal. By m-dependent, we mean that there is an integer \( m \) such that the sequences \( [N_i]_m \) and \( [N_i]_m \) are independent whenever \( i < C > m \) [note that \( m = 0 \) gives an independent process]. We will assume throughout that \( \theta \), the common univariate probability density of the noise sequence, is symmetric, continuously differentiable, and strictly positive on the entire real line. Our study is restricted to the asymptotic (large-sample-size) case and, to avoid singularity, we will consider the local or small-signal limit.

2.2. Optimum Memoryless Detection

In [4], the design of optimum memoryless detectors for the problem of Eq. (1) is considered. Because techniques similar to those of [4] will be applied in this study, the results of [4] are summarized here.

Consider memoryless detectors of the form

\[
\mathbb{P}(y; x) = \int \frac{1}{1 + e^{(x-x_0)/\Delta}} \ dx
\]

where \( g \) is a memoryless nonlinearity; \( \mathbb{P}(y; x) \) is the probability with which we accept \( H_0 \) when \( x \) is observed; and the randomization \( \gamma \) and thresholds \( \tau \) are chosen to give desired error...
probability performance. Note that the class of
detectors of the form of Eq. (2) is sufficiently
general to contain most memoryless detectors of
interest.

For large-sample-size situations, the per-
formance of detectors is commonly compared on the
basis of asymptotic relative efficiency (ARE).
For the situation of Eq. (1), the asymptotic
efficiency of a detector \( \phi_1 \) relative to another
detector \( \phi_2 \) is defined as \[ ARE(1,2) = \lim_{n \to \infty} s(n, \phi_1, \phi_2) \] (3)
where \( s(n, \phi_1, \phi_2) \) is the relative number of samples
\( \phi_1 \) requires to achieve the same power (i.e.,
probability of correct detection) that \( \phi_2 \) achieves
for sample size \( n \) when both are operating at
false-alarm probability \( \alpha \) and the signal strength
is 0. In [4], the optimum memoryless detector is
considered to be the detector of the form of
Eq. (2) which is the most efficient asymptotically
in the sense of Eq. (3). By considering a se-
quence of signal strengths \( g = K/n \) where \( K > 0 \),
it can be shown under mild restrictions that the
asymptotic efficiency of a detector \( \phi(g) \) of
the form of Eq. (2) relative to another detector
\( \phi(g_0) \) of the form of Eq. (2) is given by
\[ ARE(g, g_0) = \left( \frac{\sigma(g)}{\sigma(g_0)} \right)^2 \] (4)
where
\[ \sigma(g) = \left( \int g f(x) \right)^2 \]
is the efficacy of \( \phi(g) \); the function \( f \) is the
noise density; and the quantity \( \sigma(g)^2 \) is defined
as
\[ \sigma(g)^2 = \text{Var}(g(X_1) + 2 \Sigma \text{ Cov}(g(X_1)g(X_{j+1})) \right) \] (5)
Note that the subscript 0 denotes quantities
computed under the signal-absent hypothesis \( \Pi_0 \).

We see from Eq. (4) that the most efficient
memoryless detector of the form of Eq. (2) is
based on a nonlinearity \( g_0 \) solving
\[ g_0 = \arg\max_g \Pi_0(\phi(g)) \] (7)
In [4] it is shown under mild restrictions that
\( g_0 \) solves Eq. (7) if, and only if, \( g_0 \) (or some
constant multiple of \( g_0 \)) satisfies the integral
equation
\[ g_0(x) = \sum_{k} \int K(x,y) f_0(y) dy = \Psi_0(x) \] (8)
for all \( x \in \Xi \). Here \( \Psi_0 \) is - \( f'/f \) is the opti-
mum choice of nonlinearity for the problem in
Eq. (1) when the noise process is an independent
sequence (\( n=0 \)); and the kernel \( K(x,y) \) is given by
\[ K(x,y) = \sum_{j=1}^{m} f_{j,1} (y|x) f_{1,j} (y|x) \] (9)
where \( f_{j,1} \) is the conditional density of \( N_{j+1} \)
given \( N_j \), and vice-versa for \( f_{1,j} \).

Equation (8) is a Fredholm equation of the
second kind, and under further mild conditions,
the solution is given by
\[ g_0(x) = \Psi_0(x) - \sum_{c} \frac{(c_1, y) / (1 + c_1, y)}{w_0(x) - v_0 \Psi_0(x)} \] (10)
where equality is in the sense of uniform
convergence. Here \( \{ c, v : v=0, \ldots, \} \) is a sequence
of functions, orthogonal to respect to \( f, \) satisfying
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(x,y)| f(y) dy = \] (11)
where
\[ \lambda = \sum_{c} \frac{c_1, y / (1 + c_1, y)}{w_0^2 (1 + c_1, y)} \] (12)
and the coefficients \( c_v \) are given by
\[ c_v = \frac{\int \Psi_0 \Psi_0 f(x) f(y) dy}{\int \Psi_0 \Psi_0 f(x) f(y) dy} \] (13)
where \( \Psi_0 \) is the solution of
\[ ARE(g_0, g_0) = \sum_{c} \frac{c_1, y / c_1, y}{w_0^2 (1 + c_1, y)} \] (14)
Further details of these results can be found in [4].

2.3. Optimum Quantization for the Independent
(\( n=0 \)) Case

In this paper we will consider the subclass of
the class of memoryless detectors of Eq. (2) for
which the nonlinearity \( g \) is an \( N \)-level quan-
tizer, that is, we consider the class of all
detectors of the form
\[ g(Q; g) = \{ 1 : \ n > r \} \] (15)
where \( Q \) is an \( N \)-level quantizer; and we wish to
choose \( Q \) in an optimal way for the detection
problem of Eq. (1). Note that an \( N \)-level quan-
tizer can be parameterized by a pair \( Q = (x, q) \) where
\( x = (x_0, t_1, \ldots, t_{N-1}) \) is an ordered breakpoints
vector \( (-\infty = t_0 < t_1 < \cdots < t_{N-1} < \infty) \); \( q \in [n] \) is a levels vector; and we take
\[ Q(c) = q_k \] when \( c \in [t_{k-1}, t_k) \); \( k=1, \ldots, \) (16)
Thus the problem of optimum quantizer design is
one of performance optimization over a set of
\( (2M-1) \) real parameters.

The problem of optimum quantizer-design
problem for the independence noise \( (n=0) \) case of
Eq. (1) has been considered by Kassam in [1].
Specifically it is shown in [1] that, for this
case, if we fix the breakpoints \( q \) then an optimi-
num (asymptotically-most-efficient) choice of
the levels vector \( q \) is given by
where \( f \) is the univariate noise density. Thus, for each \( x \), the optimal choice of levels \( a_k \) can be written in terms of \( f(x) \) and the problem of choosing an optimal quantizer \( Q \) is reduced to that of choosing an optimal breakpoints vector \( a_0 \). It is shown in [1] that the optimum \( \gamma \) for this case (\( n=0 \)) must satisfy
\[
\gamma_k = (f(x_k - 1) - f(x_k))/f(x_k) \quad ; k = 1, \ldots, N (17)
\]
where \( \gamma_k \) is from Eq. (17) and the function \( f(x) \) is from Eq. (16). Thus a simultaneous solution to Eqs. (17) and (18) will yield the optimum quantizer parameters for the independent noise case of Eq. (1). Further details may be found in [1].

3. Optimum Quantization for \( n \)-Dependent Noise

3.1. The Efficacy of Quantizer-Detectors

We see from the results summarized in Section 2 that the optimum memoryless nonlinearity for detecting a constant signal in \( n \)-dependent noise \([\text{Eq. (3)}]\) is given by Eq. (10). Similarly, the optimum quantizer-detector parameters for the independent noise case (\( n=0 \)) of Eq. (1) are seen to be solutions to Eqs. (17) and (18). In this section we derive quantizer design equations analogous to those Eqs. (17) and (18) for the general case of \( n \)-dependent noise using analysis similar to that used to derive Eq. (10).

We consider the class of memoryless quantizer-detectors \( x(Q(x)); \) of the form of Eq. (15). As in Eq. (16), the quantizer is parameterized by identifying \( Q \equiv (\gamma, \delta) \) where \( \delta \) and \( \gamma \) are the breakpoints and levels of \( Q \), respectively. Adopting the optimality criterion of maximum \( \text{ARE} \), we wish to choose vectors \( \gamma \) and \( \delta \) so that the quantizer \( Q^0 = (\gamma^0, \delta^0) \) is optimum for the problem of Eq. (1). Within minor restrictions (as in [4]), the relative efficiency of one quantizer-detector \( \delta(Q(x)) \) of the form of Eq. (15) relative to another quantizer-detector \( \delta(Q'(x)) \) of the form of Eq. (15) is given by
\[
\text{ARE}(Q^0, Q^0) = \frac{\eta(Q^0)}{\eta(Q^0)} = \eta(Q^0) (19)
\]
where, as in Eq. (15), the efficacy of \( \delta(Q(x)) \) is given by
\[
\eta(Q) = (\int q \cdot f) f/\sigma_0^2 (Q), (20)
\]
and \( \sigma_0^2 \) is defined by Eq. (6). Thus the optimum \( N \)-level quantizer \( Q^0 \) will be given by
\[
Q^0 = \arg \{ \max \eta(Q) \} \quad Q \in P_N (21)
\]
where \( P_N \) is the class of all \( N \)-level quantizers

(Here we must introduce the restriction to those \( Q \) for which \( \sigma_0^2(Q) > 0 \). Without this property, Eq. (19) does not hold. Note that this is not a very restrictive limitation since we always have \( \sigma_0^2(Q) \geq 0 \), and the case \( \sigma_0^2(Q)=0 \) is of limited interest.)

Since \( Q \) is parameterized by the vectors \( \gamma \) and \( \delta \), the efficacy \( \eta(Q) \) is a function of \( 2N-1 \) real parameters (note that \( \gamma_0 \) and \( \delta_0 \) are fixed) so we may write explicitly
\[
\eta(Q) = \eta(\gamma, \delta) (22)
\]
Note from Eq. (20) that \( \eta(Q) \) is invariant to additive constants, that is \( \eta(Q+\bar{a}) = \eta(Q) \) for any constant \( \bar{a} \). Note also that \( f \) is assumed to be symmetric; thus, we may, without loss of generality, restrict our study to those quantizers \( Q \) which have zero mean under \( H_0 \); i.e., we restrict \( Q \) to satisfy
\[
E_x [Q(X)] = \int Qf = 0.
\]

It can be shown that \( \eta(\gamma, \delta) \) is given by
\[
\eta(\gamma, \delta) = (\int q \cdot f^2)^{1/2} (23)
\]
where the superscript \( T \) denotes transpose. Here the vector \( \gamma \) has components
\[
\gamma_k = [f(x_k - 1) - f(x_k)] ; k = 1, \ldots, N; (24)
\]
the matrix \( F \) is defined by
\[
F = \text{diag}[f(x_1), \ldots, f(x_N)] \quad \text{where}
\]
\[
f_k = \int f(x) \quad ; k = 1, \ldots, N; (25)
\]
and the \( M \times M \) matrix \( P \) has entries
\[
P_{k,i} = \sum_j [P_0[X_t \in (\gamma_{k-1}, \gamma_k), \gamma_{j-1} \in (\gamma_{j-1}, \gamma_j)]
\]
\[
+ P_0[X_t \in (\gamma_{j-1}, \gamma_j), \gamma_{k-1} \in (\gamma_{k-1}, \gamma_k)] \quad ; k, i = 1, \ldots, M \quad \text{where}
\]
where \( P_0 \) denotes probability under the signal-absent hypothesis \( H_0 \). Recall that \( f \) is the univariate density, \( n \) is the dependence parameter, and \( M \) is the dimension (i.e., number of levels) of the quantizer.

3.2. Results for Fixed Breakpoints - Optimum Choice of Levels

As in the independent noise case of [1], it is convenient to consider first the situation where the breakpoints vector \( \gamma \) is predetermined or fixed. Here we are free to choose only the levels, and in this case we look for optimum levels \( \delta \) by searching for a solution \( \delta \) to satisfy
\[
\delta = \arg \{ \max \eta(\gamma, \delta) \} \quad \delta \in \mathbb{R}^N
\]
To do so, we assume that the matrix \( (F + P) \) is
The positive definiteness of \( F + P \) is from Eqs. (25 and 26), respectively. Within this restriction, a necessary condition for a solution to Eq. (27) is that
\[
\text{grad } \eta(Q, a) \mid_{a=0} = 0
\]
This leads to the condition
\[
(F + P)Q^0 = 0
\]
where \( F, P, \) and \( Q \) are defined by Eqs. (24) through (26). Note that the dependence on \( Q \) of \( \Delta F, \Delta P, \) and \( Q^0 \) is implicit in Eq. (29). Equation (29) can also be shown to be a sufficient condition for \( Q^0 \) to solve Eq. (27).

Since the noise density \( f \) is assumed to have support \( (-\infty, \infty) \), the terms \( f_1, \ldots, f_N \) on the diagonal of the matrix \( F \) are all positive. Thus, \( F \) is invertible and Eq. (29) can be rewritten in a more intuitively appealing form:
\[
\begin{align*}
\Delta F_0 - F &= 0 \\
\Delta P &= 0
\end{align*}
\]
where the \( N \times N \) matrix \( F = F^{-1} \) has entries
\[
\begin{align*}
K_{k,l} &= F_{k,l} = \frac{1}{N} \\
K_{k,l} &= 1, \ldots, N
\end{align*}
\]
and \( F_{k,l} \) and \( K_{k,l} \) are from Eqs. (25) and (26), and the vector \( \Delta F_0 = F^{-1}(\Delta F) \) has components
\[
\begin{align*}
\Delta F_{k,1} &= -K_{k,1} f(x) dx = \int_{-\infty}^{\infty} f(x) dx,
\end{align*}
\]
\( k = 1, \ldots, N \)
is the optimum choice of levels for the independent noise (model) case from Eq. (17). Comparing Eq. (30) to the integral equation [Eq. (8)] of Section 2, we see the analogy between the quantized (fixed-breakpoints) case and the continuous nonlinearity case.

The positive definiteness of \( F + P \) implies that \( \eta(Q, a) \) is invertible (where \( F \) is an \( N \times N \) identity matrix). Thus we have the solution
\[
\begin{align*}
\Delta F &= 0 \\
\Delta P &= 0
\end{align*}
\]
Using the identity
\[
(L + K)^{-1} = L^{-1} - KL^{-1}K^{-1}
\]
Eq. (33) becomes
\[
\begin{align*}
\text{ARE} &= \Delta F_0 - KL^{-1}K^{-1}
\end{align*}
\]
Note that Eq. (35) is directly analogous to the continuous nonlinearity solution given by Eq. (10) of Subsection 2.2. Note also that Eq. (35)
\[
\text{ARE} = \frac{\eta_0^2(Q, Q)}{\eta(Q, Q)}
\]
where \( Q \) is such that \( F + P \) is invertible, Eqs. (23), (37), and (38) imply that
\[
\begin{align*}
\text{ARE} &= \frac{[\eta_0(Q, a)]^2}{[\eta_0(Q, a)]^2}
\end{align*}
\]
Specifically, the improvement gained by using \( Q \) over the independent-noise levels \( \eta_0 = \eta^{(0)}(\Delta F) \) is given by
\[
\begin{align*}
\text{ARE} &= \frac{[\eta_0(Q, a)]^2}{[\eta_0(Q, a)]^2}
\end{align*}
\]
If the breakpoints \( a \) are chosen so that they partition the real line into equiprobability segments (under \( H_a \)), then \( F = \frac{1}{N} \), and we have

\[\text{ARE}\] could be derived by applying the Schwarz inequality to the efficiency \( \eta(Q) \). The approach here is used because of the analogy to that for the continuous nonlinearity case.

Even if \( F + P \) is not positive definite, Eq. (30) still has a solution given by
\[
\begin{align*}
\Delta F &= a - (\Delta F + \Delta P)
\end{align*}
\]
where the superscript \( + \) denotes the generalized inverse. Equation (36) can be verified by direct substitution into Eq. (30) if we note that the matrix \( (F + P)^+ \) is the projection onto the range space of \( (F + P) \). Note however that, if \( F + P \) is not positive definite, then the necessity of Eq. (28) is not assured and the expression of Eq. (29) cannot be used for all quantizers \( Q \). (Since \( \eta^2(Q) \) will equal zero for some \( Q \)).


Choosing Optimum Breakpoints

We turn now to the problem of choosing an optimal set of breakpoints \( \xi \). An overall optimum choice of quantizer \( Q_0 = (\xi, \xi) \) will be one for which \( \eta(\xi, \xi) \) is a maximum over all possible choices of \( \xi \) and \( \xi \). From the results of Subsection 3.2 we have that, for fixed \( \xi \), the optimal choice of levels is made by choosing \( Q_0 \) from Eq. (33) corresponding to \( \xi \). Writing explicitly \( \eta(\xi, \xi) \) to emphasize the dependence on \( \xi \) of the solution from Eq. (33), we have

\[
\max \eta(\xi, \xi) = \eta(\xi, \xi)(t) \quad (43)
\]

Thus, as in the independent noise case of Subsection 2.3, the problem of maximizing \( \eta(\xi, \xi) \) over \( \xi \) can be reduced to that of maximizing \( \eta(\xi, \xi)(t) \) over \( \xi \); that is, we look for \( \xi \) to solve

\[
\xi = \arg\max \eta(\xi, \xi)(t) \quad (44)
\]

and the overall optimal quantizer will be given by \( Q_0 = (\xi, \xi) \).

To look for solutions to Eq. (44), we restrict the noise process to be such that the matrix \( \Sigma^{ij} \) is positive definite for every choice of \( \xi \). For this case, we have from Eq. (37) that

\[
\eta(\xi, \xi)(t) = (\xi + t)^T (\xi + t) + P(t) \quad (45)
\]

A necessary condition for \( \xi \) to maximize (45) is that

\[
\frac{\partial \eta(\xi, \xi)(t)}{\partial \xi} = 0 \quad (46)
\]

Using the identity

\[
\frac{\partial (\xi + t)^T (\xi + t) + P(t)}{\partial \xi} = 2(\xi + t) + P(t) \quad (47)
\]

Eq. (46) can be shown to hold if

\[
\xi_{0}(t) = \frac{1}{2} L \xi(t) \quad (48)
\]

where the vector \( \xi_{0}(t) = (\xi_{0}(t)^T, \ldots, \xi_{0}(t)^T, \ldots, \xi_{0}(t)^T)^T \), the function \( \xi = f(t) \), and the \((M-1)X \) matrix \( L \) has entries

\[
L_{k,l} = \frac{\xi_{0}(k+1,t) + \xi_{0}(k-1,t)}{2} + \sum_{p=0}^{M-1} \frac{\partial \xi_{0}(t)^T}{\partial \xi_{0}(t)} \quad (49)
\]

for \( k = 1, \ldots, (M-1) \) and \( l = 1, \ldots, M \).

Here \( L \) is the Kronecker delta and the matrix \( P \) is defined from Eq. (26).

Thus, combining Eqs. (32) and (47), an overall optimal quantizer \( Q_0 = (\xi, \xi) \) will be a solution to the equations

\[
\xi(t) = \frac{1}{2} L \xi(t) \quad (49)
\]

and

\[
\xi = f(t) \quad (50)
\]

where the quantities \( \xi(t) \) and \( \xi(t) \) are computed for \( t = \xi(t) \) from Eqs. (33) and (48). That is, under the above assumptions, Eqs. (49) and (50) are necessary conditions that must be satisfied by an optimum (most efficient) quantizer for a memoryless decision between the hypothesis pairs of Eq. (1). Given that \( \xi(t) \) is optimal and \( \xi(t) \) is positive definite, Eq. (50) is sufficient for \( \xi(t) \) to be optimal; however, the sufficiency of Eq. (49) can be determined analytically only for special cases, and a numerical test must be used in the general case.

Note that for the independent-noise (no) case, the entries of the matrix \( L \) become

\[
L_{k,l} = \begin{cases} 1, & k = l, \ldots, (M-1) \ \text{or} \ k = 1, \ldots, (M-1) \text{ and } \ l = 1, \ldots, M \end{cases}
\]

and the matrix \( P \) is identically zero. Thus, for this case, Eqs. (49) and (50) reduce to Eqs. (17) and (18) so that Eqs. (49) and (50) agree with the previously established results for this case.

4. Example — 4-Level Quantization

Example 1 — Gaussian Noise

To illustrate the results of Section 3, we consider the particular problem of designing 4-level quantizers \((M=4)\) for use in detecting known signals in \(\alpha\)-dependent Gaussian noise.

Specifically, we assume that the noise process \([N(t), t = 1, \ldots, \alpha] \) arises from uniform sampling of a stationary Gaussian process \(W(t)\). We assume that \(W(t)\) has zero mean, unit variance, and autocorrelation \(\rho(t) = \rho(t)\) given by

\[
\rho(t) = \begin{cases} 1 - |\tau| / D, & |\tau| \leq D \\ 0, & |\tau| > D \end{cases}
\]

where \(D > 0\). The samples are taken at intervals of length \(D/\alpha\), and, at this sampling rate, \([N(t), t = 1, \ldots, \alpha] \) is \(\alpha\)-dependent. For this case, the elements of the matrix \(L\) of Eq. (31) are given by

\[
L_{k,l} = \begin{cases} \frac{1}{2} (k + 1, k - 1), & k = 1, \ldots, \alpha - 1 \\ 1, & k = 1, \alpha \end{cases}
\]

and

\[
\xi(t) = \frac{1}{2} L \xi(t) \quad (49)
\]

and

\[
\xi = f(t) \quad (50)
\]

where the quantities \( L_{k,l} \) and \( \xi(t) \) are computed for \( t = \xi(t) \) from Eqs. (33) and (48). That is, under the above assumptions, Eqs. (49) and (50) are necessary conditions that must be satisfied by an optimum (most efficient) quantizer for a memoryless decision between the hypothesis pairs of Eq. (1). Given that \( \xi(t) \) is optimal and \( \xi(t) \) is positive definite, Eq. (50) is sufficient for \( \xi(t) \) to be optimal; however, the sufficiency of Eq. (49) can be determined analytically only for special cases, and a numerical test must be used in the general case.

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\]

and the matrix \( P \) is identically zero. Thus, for this case, Eqs. (49) and (50) reduce to Eqs. (17) and (18) so that Eqs. (49) and (50) agree with the previously established results for this case.

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\[
\rho(t) = \begin{cases} 1 - |\tau| / D, & |\tau| \leq D \\ 0, & |\tau| > D \end{cases}
\]

where \(D > 0\). The samples are taken at intervals of length \(D/\alpha\), and, at this sampling rate, \([N(t), t = 1, \ldots, \alpha] \) is \(\alpha\)-dependent. For this case, the elements of the matrix \(L\) of Eq. (31) are given by
Here, the optimum quantizer $Q^0(q_0,q_0)$ for the case $\kappa_{(1)}=1$ (i.e., 4-level quantization) is characterized by
\[ -q_1^0 = q_4^0 = 0; \quad -q_2^0 = q_3^0 > 0 \]
and
\[ -q_1^0 = q_4^0 = 0; \quad -q_2^0 = q_3^0 > 0 \]
Since $\eta(Q)$ is independent of scale (i.e., $\eta(q_0)$ = $\eta(q)$ for $|\alpha| \neq 0$), we may thus characterize $Q^0$ completely by specifying the upper breakpoint $t^0$ and the levels ratio $(q_4/q_3)$. Table I gives values of these parameters for several values of the dependence parameter $m$. Note that $m=0$ gives the independent case. Also given in Table I are values of $ARE(Q_0,Q_0)$ and $ARE(Q_0,\xi,d.)$ where $Q_0$ denotes the $m=0$ quantizer and $\xi,d.$ denotes the linear detector (i.e., Eq. (2) with $g(x) = x$), which is the optimum memoryless detector for this case [6]. $ARE(Q_0,\xi,d.)$ is a measure of the improvement gained by these techniques over the system designed by merely ignoring the dependence, and $ARE(Q_0,\xi,d.)$ is a measure of the degradation in performance due to quantization.

**Example 2 - Cauchy Noise**

As a second example consider the noise process $\{\tau(t_i)\}_{i=1}^m$ where
\[ \tau(t) = \tan(\arctan(x/\sqrt{2})/2) \] and $\{x_i\}_{i=1}^m$ is a Gaussian process identical to the process $\{x_i\}_{i=1}^m$ of Example 1. This will yield a Cauchy noise process with univariate density $f(x)=[\pi(1+x^2)]^{-1}$. The optimum 4-level quantizer for $\{\tau(t_i)\}_{i=1}^m$ will also satisfy Eq. (50) and its parameters are given in Table II. Note that, here, there are two equivalent optimum quantizers for the $m=0$ case. These are denoted by $Q_0^0$ and $Q_2^0$, respectively, and performance of the quantizer $Q_0$ relative to each of these is given in Table II.

Note that $q_0^0 < 0$ for some values of $m$, this is consistent with results for the unquantized memoryless case for this example as discussed in [6].

### Acknowledgments

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### References


### Table I. 4-Level Quantizer Parameters for Gaussian Noise

<table>
<thead>
<tr>
<th>m</th>
<th>$c_3^0$</th>
<th>$(q_0^0/q_3^0)$</th>
<th>$ARE(Q_0^0,\xi,d.)$</th>
<th>$ARE(Q_0^0,\xi,d.)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.982</td>
<td>3.34</td>
<td>1.00</td>
<td>.882</td>
</tr>
<tr>
<td>1</td>
<td>1.03</td>
<td>3.40</td>
<td>1.000</td>
<td>.935</td>
</tr>
<tr>
<td>2</td>
<td>1.09</td>
<td>3.46</td>
<td>1.002</td>
<td>.953</td>
</tr>
<tr>
<td>3</td>
<td>1.19</td>
<td>3.44</td>
<td>1.005</td>
<td>.972</td>
</tr>
<tr>
<td>4</td>
<td>1.24</td>
<td>3.40</td>
<td>1.006</td>
<td>.981</td>
</tr>
<tr>
<td>5</td>
<td>1.26</td>
<td>3.38</td>
<td>1.007</td>
<td>.983</td>
</tr>
<tr>
<td>6</td>
<td>1.28</td>
<td>3.38</td>
<td>1.007</td>
<td>.986</td>
</tr>
</tbody>
</table>

### Table II. 4-Level Quantizer Parameters for a Cauchy Noise Process

<table>
<thead>
<tr>
<th>m</th>
<th>$c_3^0$</th>
<th>$(q_0^0/q_3^0)$</th>
<th>$ARE(Q_0^0,\xi,d.)$</th>
<th>$ARE(Q_0^0,\xi,d.)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.254</td>
<td>2.92</td>
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<td>1.00</td>
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<tr>
<td>2</td>
<td>3.94</td>
<td>.342</td>
<td>1.000</td>
<td>1.00</td>
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<td>-2.38</td>
<td>2.53</td>
<td>2.14</td>
</tr>
<tr>
<td>2</td>
<td>2.13</td>
<td>-0.0802</td>
<td>2.90</td>
<td>2.31</td>
</tr>
<tr>
<td>5</td>
<td>2.12</td>
<td>-2.37</td>
<td>3.50</td>
<td>2.63</td>
</tr>
<tr>
<td>10</td>
<td>1.88</td>
<td>-2.27</td>
<td>3.81</td>
<td>2.80</td>
</tr>
</tbody>
</table>
The problem of optimum data quantization for memoryless signal-detection systems operating in m-dependent noise environments is considered. The case where quantizer breakpoints are fixed or predetermined is considered first, and, for this case, existing results for general (unquantized) memoryless detection are modified to yield necessary and sufficient conditions for
quantizer optimization in terms of asymptotic efficiency. Necessary conditions are also established for the optimum (asymptotically efficient) selection of quantizer breakpoints, and expressions are presented for the comparison of quantizer-detector performance on the basis of asymptotic relative efficiency.